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## LETTER TO THE EDITOR

# The size function in two-dimensional bond percolation: a series analysis

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**Abstract.** We extend the known series expansion coefficients for the mean cluster size in two-dimensional bond percolation and use them to accurately approximate the size function for all probabilities less than the critical probability.

The mean number of bonds per cluster  $S(p)$  is an important descriptor in percolation theory (for reviews, see Essam (1972, 1980) and Stauffer (1985)). It is generally believed that, as the bond occupation probability  $p$  increases from zero,  $S(p)$  diverges at some critical probability  $p_c$  according to

$$S(p) = W(p)/(1-p/p_c)^\gamma \quad 0 \leq p < p_c$$

where the 'reduced size function'  $W(p)$  is finite at  $p = p_c$  and less singular than  $S(p)$  there. Sykes and Essam (1963) have deduced, from very modest assumptions, that for the two-dimensional square lattice  $p_c = 1/2$ , while for the two-dimensional triangular lattice  $p_c = 2 \sin(\pi/18)$ . It has also been conjectured (den Nijs 1979, Nienhuis *et al* 1980, Nienhuis 1982) that for all two-dimensional lattice percolation problems  $\gamma = 43/18$ . If these results are accepted (and we shall do so here), then the unknown features of  $S(p)$  are all contained within the reduced function  $W(p)$ . This letter uses series extrapolation methods to find an accurate approximation for  $W(p)$ .

Let us denote the expansion coefficients of  $S(p)$  by  $s_i$  so that  $S(p)$  has the formal power series

$$S(p) = \sum_{i=0}^{\infty} s_i p^i.$$

Coefficients up to and including  $s_{14}$  for the square lattice and  $s_{10}$  for the triangular lattice were computed by Sykes and Glen (1976). We have extended both of these series by two terms. To find these coefficients for the square lattice size function we first enumerated all connected clusters of 16 or fewer bonds, classified by the number of perimeter bonds. From these data it is easy to find the coefficients  $s_0$  through  $s_{15}$ , and by a trick (Sykes and Glen 1976) it is also possible to find  $s_{16}$ . The enumeration was performed on a Cray X-MP computer, using a variant of Martin's (1974) backtrack algorithm, and required 27 CPU hours. Similarly, the triangular lattice enumeration examined all clusters of 12 or fewer bonds, produced coefficients  $s_0$  through  $s_{12}$ , and required 82 CPU hours on a VAX 8600. The expansion coefficients are given in table 1.

With the series coefficients in hand, one could approach the problem using the standard techniques of series analysis in statistical mechanics, which focus on the singularity at  $p_c$ . This is not the approach taken here. Because this singularity is well understood, while the reduced function  $W(p)$  is not, we choose instead to study  $W(p)$ .

Table 1. The expansion coefficients  $s_i$ .

$i$	$s_i$ (square)	$s_i$ (triangular)
0	1	1
1	6	10
2	18	46
3	48	186
4	126	706
5	300	2 568
6	762	9 004
7	1 668	30 894
8	4 216	103 832
9	8 668	343 006
10	21 988	1 123 770
11	43 058	3 623 234
12	110 832	11 630 150
13	202 432	
14	561 020	
15	875 382	
16	2881 286	

(The expansion coefficients for  $W(p)$  are readily obtained from the expansion coefficients for  $S(p)$ .) Because  $W(p)$  is less singular than  $S(p)$ , one expects that it will be easier to approximate, and our experience supports this expectation. Neither rational (Padé) nor differential approximants (see Baker and Graves-Morris 1981, Fisher and Au-Yang 1979) suggest a singularity in  $W(p)$  for positive  $p$ . Furthermore, rational and differential approximants give numerically similar values for  $W(p_c)$ , another good indication that  $W(p)$  is smooth or only slightly singular.

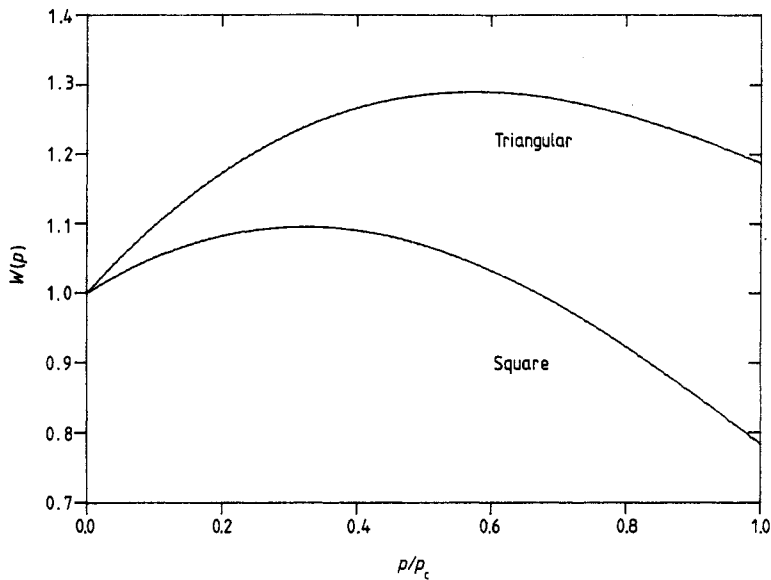
To investigate the square lattice reduced size function, we generated all those inhomogeneous differential approximants (IDA) (Fisher and Au-Yang 1979, Hunter and Baker 1979) which used the coefficients of  $W(p)$  up to and including order 14, 15 or 16. The resulting 409 approximants include Padé and Dlog Padé approximants, which are special cases of the IDA. These approximants are not all created equal—for example the Euler invariant approximants of form  $[\emptyset/L; L+2]$  or  $[L/L; L+2]$  are expected to be the most accurate approximants, while the truncated polynomials of form  $[J/0; \emptyset]$  are known to be inaccurate. Thus it is not surprising that several approximants predict implausibly high or impossibly negative values for  $W(p_c)$ . But it is surprising—and heartening—that a total of 269 approximants predict values for  $W(p_c)$  which fall in a tight cluster between 0.7 and 0.8. Furthermore, this group includes the vast majority of the approximants which are expected *a priori* to be accurate. The mean value of  $W(p_c)$  for this cluster is 0.784, with standard deviation 0.009. The simple Padé approximant  $[8/3; \emptyset]$  predicts a value for  $W(p_c)$  very close to this mean, and in fact it differs only minimally from other good approximants throughout the range  $0 \leq p \leq p_c$ . This approximant is (coefficient values are given in table 2)

$$\bar{W}_{\text{sq}}(p) = \frac{1 + a_1 p + \dots + a_8 p^8}{1 + b_1 p + b_2 p^2 + b_3 p^3}$$

and we believe that it approximates  $W(p)$  with an accuracy of  $\pm 0.02$  at  $p_c$  and with even higher accuracy at smaller values of  $p$ . This approximant is plotted in figure 1.

**Table 2.** Coefficients for the rational approximants  $\bar{W}_{sq}(p)$  and  $\bar{W}_{tri}(p)$ .

Square lattice	
$a_1 = 6.680\ 135\ 490$	$b_1 = 5.457\ 913\ 268$
$a_2 = 12.552\ 841\ 83$	$b_2 = 9.912\ 923\ 142$
$a_3 = -3.591\ 448\ 054$	$b_3 = 6.198\ 442\ 597$
$a_4 = -26.603\ 351\ 89$	
$a_5 = -13.265\ 056\ 83$	
$a_6 = 25.244\ 496\ 65$	
$a_7 = 6.959\ 944\ 259$	
$a_8 = 17.027\ 174\ 71$	
Triangular lattice	
$a_1 = 7.904\ 205\ 291$	$b_1 = 4.782\ 736\ 702$
$a_2 = 14.831\ 958\ 44$	$b_2 = 8.933\ 983\ 888$
$a_3 = -22.447\ 700\ 30$	$b_3 = -9.136\ 096\ 944$
$a_4 = -88.697\ 715\ 55$	$b_4 = 5.276\ 250\ 594$
$a_5 = 153.544\ 621\ 3$	$b_5 = 0.351\ 593\ 4692$
	$b_6 = 2.016\ 807\ 389$



**Figure 1.** The functions  $\bar{W}_{sq}(p)$  and  $\bar{W}_{tri}(p)$  which model the reduced size function  $W(p)$  for bond percolation on the square and triangular lattices respectively.

The investigation for the triangular lattice followed the pattern established in the square lattice. All approximants using coefficients up to and including 10, 11 or 12 terms were generated. Of these 235 approximants, 188 have  $W(p_c)$  between 1.15 and 1.25. This cluster of good approximants has a mean  $W(p_c)$  equal to 1.187 with standard deviation 0.005. A rational approximant which falls in the centre of the cluster and which can be used as a model for the true  $W(p)$  is  $[5/6; \emptyset]$ , namely

$$\bar{W}_{tri}(p) = \frac{1 + a_1 p + \dots + a_5 p^5}{1 + b_1 p + \dots + b_6 p^6}$$

which approximates  $W(p)$  with an accuracy of  $\pm 0.01$  at  $p_c$  and with more accuracy at smaller values of  $p$ .

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